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Parasupersymmetry of Non-Linear and Isotropic Oscillator on Constant Curvature

H. Rahbar · M.R. Pahlavani · J. Sadeghi · H. Moayyeri

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Abstract In this paper we discuss the one-dimensional non-linear harmonic oscillator and isotropic oscillator in positive curvature. We will see that in the special case two different oscillator have a same behavior. Finally, by using the parasupersymmetry algebra we obtain the partner potential and superpotential. These superpotential lead us to obtain the corresponding superchares.

Keywords Non-linear oscillator · Parasupersymmetry · Supercharges

1 Introduction

Nonlinear oscillator system have an application to different aspects of physics as atmospheric physics, condensed matter, non-linear optics to electronics, plasma physics, biophysics, evolutionary biology [1–5]. The mass in one dimensional non-linear oscillator depend to position as $m = (1 + \lambda x^2)^{-1}$ [6–10].

And also the non-linear differential equation is,

$$(1 + \lambda x^2)\ddot{x} - (\lambda x)\dot{x}^2 + \alpha^2 x = 0, \quad \lambda > 0 \tag{1}$$

where

$$x = A \sin(\omega t + \gamma)$$

H. Rahbar (✉) · M.R. Pahlavani · J. Sadeghi · H. Moayyeri
Sciences Faculty, Department of Physics, Mazandaran University, P.O. Box 47415-416, Babolsar, Iran
e-mail: h.rahbar@umz.ac.ir

M.R. Pahlavani
e-mail: m.pahlavani@umz.ac.ir

J. Sadeghi
e-mail: pouriya@ipm.ir

H. Moayyeri
e-mail: h.moayyeri@umz.ac.ir

and frequency and amplitude is

$$\omega^2 = \frac{\alpha^2}{1+\lambda A^2}.$$

The nonlinear equation (1) is therefore an interesting example of a system with nonlinear quasiharmonic oscillations [11].

Also the accidental degeneracy in the space of constant curvature has interested many researchers. The isotropic oscillator in positive curvature caused by an additional integral of motion. However, in contrast with the flat the integrals of motion for isotropic oscillator do not simple form the algebra as relevant commutators. The group hidden symmetry with accidental degeneracy exist in isotropic oscillator in positive curvature system. In that case the non-linear oscillator can be related to the isotropic oscillator with constant curvature in special case, and also is used information from later oscillator to obtain the energy spectrum. Also we take some advantage from isotropic oscillator in constant curvature and study the supersymmetry aspect of non-linear oscillator. Note that the supersymmetry quantum mechanic (SSQM) and the concept of shape invariance have proved very useful for generating exactly solvable potential. The SUSY is an interesting symmetry which transforms bosons into fermions and vice-versa. The SSQM Quantum theory provide realization of graded Lie algebra (GLA) [12].

The natural generalization of SUSY so-called parasupersymmetry (PSUSY) which is a symmetry between bosonic and parafermionic degrees of freedom. Also, it plays a similarly important role as supersymmetry in the description of nature. The PSUSY structure depend on the number parafermions (P) that can occupy the same state.

Supersymmetric quantum mechanics in case of $P=1$ was studied by Witten [13, 14]. The generalization of this to the $P=2$ case was done by Rubakov, Spiridonov [15], Beckers and Debergh [16]. Then formalism algebra generalized by *Toshiaki Tanaka* for the PSUSY, they constructed second-order and third order parafermionic algebra and multiplication law [17, 18]. Also *Rubakov-Spiridonov* generalized PSUSY algebra to order 3 [19].

The issue generalizing the problem of isotropic oscillator for the space of constant curvature with the use of the conformally flat metric in the classical mechanics has obviously been solved for the first time [23]. In that case by using the spherical coordinates on constant positive curvature, they obtained energy spectrum and wave function and also discussed the supersymmetry aspect of this system.

This paper organize as follows: Section 2 introduce about one-dimensional model of a Quantum Nonlinear Harmonic Oscillator, in Sect. 3 comparing special case of Isotropic Oscillator on the constant curvature, in Sect. 4 we use the factorization method for Nonlinear Oscillator and finally in Sect. 5 we find supercharges, super-potentials, new potential and partner Hamiltonian for this system.

2 One-Dimensional Model of a Quantum Non-Linear Harmonic Oscillator

We are going to consider the hamiltonian formalism for the one dimensional non-linear oscillator and discuss the corresponding Schrödinger equation [6–8]. So the hamiltonian function is given by,

$$H = (1+\lambda x^2)\frac{p^2}{2} + V(x) = \frac{1}{2}(\sqrt{1+\lambda x^2}\,p)^2 + V(x). \tag{2}$$

It is important to remark that in the space $L^2(\Re, d\mu)$ of square integrable function in the real line the adjoint of the differential operator $\sqrt{1+\lambda x^2}\frac{\partial}{\partial x}$ is precisely the opposite of such operator. The only invariant measures are the multiples of $d\mu = (1+\lambda x^2)^{-\frac{1}{2}}dx$. Therefore the linear operator ($\hbar = 1$) is

$$P = -i\sqrt{1+\lambda x^2}\frac{d}{dx}.$$

So the quantum Hamiltonian operator turns out to be,

$$\hat{H} = \frac{1}{2}P^2 + V(x) = -\frac{1}{2}(1+\lambda x^2)\frac{\partial^2}{\partial x^2} - \frac{1}{2}\lambda x\frac{\partial}{\partial x} + V(x). \tag{3}$$

In the case of the non-linear harmonic oscillator we have,

$$H_0 = \frac{1}{2}\left[(1+\lambda x^2)p_x^2 + \frac{\alpha^2 x^2}{1+\lambda x^2}\right], \tag{4}$$

and

$$\hat{H}_0 = \frac{1}{2}\left[-(1+\lambda x^2)\frac{d^2}{dx^2} - \lambda x\frac{d}{dx} + \frac{\alpha^2 x^2}{1+\lambda x^2}\right]. \tag{5}$$

The corresponding Schrödinger equation for the case of $\lambda = -1$ is given by,

$$\frac{1}{2}\left[-(1-x^2)\frac{d^2}{dx^2} + x\frac{d}{dx} + \frac{\alpha^2 x^2}{1-x^2}\right]\psi = E'\psi \tag{6}$$

and

$$\hat{H}_o\psi = E'\psi, \tag{7}$$

where E' is energy spectrum for the non-linear oscillator system.

In order to obtain this energy we compare this system with the isotropic oscillator in positive constant curvature. So, in the following section we review the isotropic oscillator on the constant curvature [24] and then we compare with non-linear oscillator. This connection give us motivation to obtain some parasupersymmetry generator algebra and supercharges.

3 Isotropic Oscillator on the Constant Curvature

As we know the three-dimensional space of constant positive curvature can be realized geometrically on the three-dimensional sphere S_3 of the radius R, imbedded into the four-dimensional Euclidean space,

$${q_0}^2 + q_i q_i = R^2.$$

Note that the relation between the coordinates x_i in the tangent space and q_μ ($\mu = 0, 1, 2, 3$) is given by,

$$q_i = \frac{x_i}{\sqrt{1+\frac{r^2}{R^2}}}, \qquad q_0 = \frac{R}{\sqrt{1+\frac{r^2}{R^2}}}, \tag{8}$$

where the coordinates q_i change in the region $q_i q_i \le R^2$.

Now we are going to write the general form of isotropic oscillator potential in space of constant curvature. By using the $r^2 = {x_1}^2 + {x_2}^2 + {x_3}^2$ and (8) the following potential,

$$V(r) = \frac{\mu\omega^2}{2} r^2,$$

change to,

$$V(r) = V(q) = \frac{1}{2}\mu\omega^2 \frac{q^2}{1 - \frac{q^2}{R^2}}. \tag{9}$$

In the spherical system of coordinates,

$$\begin{aligned} q_1 &= R\sin\chi\sin\theta\cos\phi, \qquad q_2 = R\sin\chi\sin\theta\sin\phi, \\ q_3 &= R\sin\chi\cos\theta, \qquad q_0 = R\cos\chi, \end{aligned} \tag{10}$$

where $0 \le \chi < \pi$, $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$.

So, the oscillator potential in spherical system is,

$$V(\chi) = \frac{\mu\omega^2 R^2}{2}\tan^2\chi. \tag{11}$$

In order to solve the Schrödinger equation, we need to write the corresponding equation with potential (11) on constant curvature as follow ($\hbar = 1$),

$$\left[-\frac{1}{2\mu}\Delta + V\right]\Psi = E\Psi, \tag{12}$$

where Δ is the Laplace-Beltrami operator and given by,

$$\Delta = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\sqrt{g}g^{ik}\frac{\partial}{\partial x^k} \tag{13}$$

so the metric is,

$$ds^2 = g_{ik}dx^i dx^k,$$

where $g = \det(g_{ik})$, $g^{ik} = (g_{ik})^{-1}$ and $r^2 = x_i x_i$ $(i, k = 1, 2, 3)$. By using (10), (11), (12) and (13) one can obtain the Schrödinger equation (12) as the following,

$$\left\{\left(\frac{-1}{2\mu R^2}\right)\frac{1}{\sin^2\chi}\frac{\partial}{\partial\chi}\sin^2\chi\frac{\partial}{\partial\chi} + \frac{1}{2\mu R^2}\frac{l(l+1)}{\sin^2\chi} + \frac{\mu\omega^2 R^2}{2}\tan^2\chi\right\}Z(\chi) = EZ(\chi) \tag{14}$$

or

$$HZ = EZ,$$

where E is a energy spectrum for the isotropic oscillator on the constant curvature. This energy is [20–22],

$$E_N = \frac{1}{2\mu}\left[\frac{(N+1)(N+3)}{R^2} + \frac{2\nu}{R^2}\left(N + \frac{3}{2}\right)\right]. \tag{15}$$

In flat space $R \to \infty$ the energy spectrum will be as,

$$E = \left(N + \frac{3}{2}\right)\omega. \tag{16}$$

By comparing the non-linear oscillator with isotropic oscillator in constant curvature is obtained,

$$l(l+1) = \alpha^2 - 1$$

and

$$\mu R^2 = \frac{\alpha}{\omega},$$

where l, R and μ are parameters in oscillator with constant curvature and α is parameter from non-linear oscillator, the relation between two functions $Z(\chi)$ and $\psi(\chi)$ will be,

$$Z(\chi) = \frac{\varepsilon}{\sqrt{1-\cos 2\chi}}\psi(\chi), \tag{17}$$

where ε is normalization factor and $\cos 2\chi \equiv x$. So, one can obtain the following relation for the corresponding hamiltonian,

$$\frac{\alpha\sqrt{1-x}}{4\omega} H\left(\frac{1}{\sqrt{1-x}}\right) - \frac{3}{8}\alpha^2 + \frac{1}{4} = \hat{H}_0, \tag{18}$$

where $\hat{H}_0$ and H with energy spectrums E' and E are associated with the non-linear oscillator and isotropic oscillator in positive constant curvature respectively. We will obtain the relation between the energy spectrums of two oscillators in two different space,

$$E = \frac{4\omega}{\alpha}E' + \frac{3}{2}\omega\alpha - \frac{\omega}{\alpha}. \tag{19}$$

This issue guides one to the suggestion that in the special case nonlinear oscillator on the flat space maybe similar behavior to the isotropic oscillator on the curved space. So, this results lead one to obtain the superpotanial and supercharges for the non-linear oscillator.

4 The Factorization Method for Nonlinear Oscillator

Now back to one-dimensional nonlinear harmonic oscillator on the flat space $\lambda = -1$. Let us try to determine a function $W_1(x)$ which is called super-potential function. In order to factorize. The corresponding Schrödinger equation one can obtain A_1 and its adjoint operator A_1^+ which are given by,

$$A_1 = \frac{1}{\sqrt{2}}\left(\sqrt{1-x^2}\frac{d}{dx} + W_1(x)\right),$$
$$A_1^+ = \frac{1}{\sqrt{2}}\left(-\sqrt{1-x^2}\frac{d}{dx} + W_1(x)\right),$$

where

$$\hat{H}_0 = A_1^+ A_1 = \frac{1}{2}\left[-\sqrt{1-x^2}\frac{d}{dx} + W_1(x)\right]\left[\sqrt{1-x^2}\frac{d}{dx} + W_1(x)\right]. \tag{20}$$

Here, the super-potential function W_1 must be satisfied by following Riccati type of differential equation,

$$\sqrt{1-x^2}W_1' - W_1^2 + 2V_0 = 0. \tag{21}$$

Also, we can define a new quantum hamiltonian operator as,

$$\hat{H}_1 = A_1 A_1^+ = \frac{1}{2}\left[\sqrt{1-x^2}\frac{d}{dx} + W_1(x)\right]\left[-\sqrt{1-x^2}\frac{d}{dx} + W_1(x)\right], \tag{22}$$

which is called the partner hamiltonian. The new potential V_1 can be written is in terms of super-potential W_1,

$$V_1 = \frac{1}{2}(\sqrt{1-x^2}W_1' + W_1^2). \tag{23}$$

We factorize (5) such that (20) can be satisfied. So, one can obtain the following linear operator in $L^2(\Re, d\mu)$,

$$\begin{aligned} A_1 &= \frac{1}{\sqrt{2}}\left(\sqrt{1-x^2}\frac{d}{dx} + \frac{\beta x}{\sqrt{1-x^2}}\right), \\ A_1^+ &= \frac{1}{\sqrt{2}}\left(-\sqrt{1-x^2}\frac{d}{dx} + \frac{\beta x}{\sqrt{1-x^2}}\right), \end{aligned} \tag{24}$$

and

$$A_1^+ A_1 = -\frac{1}{2}(1-x^2)\frac{d^2}{dx^2} + \frac{1}{2}x\frac{d}{dx} + \frac{1}{2}\beta(\beta-1)\left(\frac{x^2}{1-x^2}\right) - \frac{1}{2}\beta, \tag{25}$$

where β is new parameter and play important role to the shape invariance condition. Also note that the super-potential is,

$$W_1(x) = \frac{\beta x}{\sqrt{1-x^2}} \tag{26}$$

by comparing (25) with (5), we conclude that the Hamiltonian $\hat{H}_0' = \hat{H}_0 - \frac{1}{2}\beta$ admits to the following relation,

$$\hat{H}_0' = A_1^+ A_1, \tag{27}$$

where parameters α and β are related by,

$$\beta = \frac{1}{2}(1 + \sqrt{1+4\alpha^2}). \tag{28}$$

Now partner hamiltonian $\hat{H}_1' = A_1 A_1^+$ obtained,

$$\hat{H}_1' = A_1 A_1^+ = -\frac{1}{2}(1-x^2)\frac{d^2}{dx^2} + \frac{1}{2}x\frac{d}{dx} + \frac{1}{2}\beta(\beta+1)\left(\frac{x^2}{1-x^2}\right) + \frac{1}{2}\beta. \tag{29}$$

In general case a quantum hamiltonian $\hat{H}_0(\alpha)$ admits a factorization form such that the partner hamiltonian $\hat{H}_1(\alpha)$ is the same as $\hat{H}_0(\alpha)$. But for the different values of the parameter α it is usually said that there is shape invariance

$$\hat{H}_1(\alpha) = \hat{H}_0(\alpha_1) + R(\alpha_1), \tag{30}$$

where $\alpha_1 = f(\alpha)$ and $R(\alpha)$ are constant.

Finally the shape invariance condition lead us to obtain the supercharges which is completely satisfied with parasupersymmetry algebra.

5 Second Order of Parasupersymmetry and Supercharges

PSUSY for $P = 2$ is

$$\begin{aligned} \hat{H}'_0 &= \frac{1}{2}Q_1^- Q_1^+ + c, \\ \hat{H}'_1 &= \frac{1}{2}Q_1^+ Q_1^- + c = \frac{1}{2}Q_2^- Q_2^+ - c, \\ \hat{H}'_2 &= \frac{1}{2}Q_2^+ Q_2^- - c, \end{aligned} \tag{31}$$

where c is constant. If $c = 0$ then

$$\begin{aligned} Q_1^- &= \sqrt{2}A_1^+ = -\sqrt{1-x^2}\frac{d}{dx} + \frac{\beta x}{\sqrt{1-x^2}}, \\ Q_1^+ &= \sqrt{2}A_1 = \sqrt{1-x^2}\frac{d}{dx} + \frac{\beta x}{\sqrt{1-x^2}}, \\ Q_2^- &= \sqrt{2}A_2^+ = -\sqrt{2}A_1 = -\sqrt{1-x^2}\frac{d}{dx} - \frac{\beta x}{\sqrt{1-x^2}}, \\ Q_2^+ &= \sqrt{2}A_2 = -\sqrt{2}A_1^+ = \sqrt{1-x^2}\frac{d}{dx} + \frac{\beta x}{\sqrt{1-x^2}} \end{aligned}$$

and is noted that,

$$\begin{aligned} \hat{H}'_k &= \frac{1}{2}P^2 + V'_k(x), \\ Q_j^\pm &= \pm\sqrt{1-x^2}\frac{d}{dx} + W_j(x) \end{aligned} \tag{32}$$

where $(k = 0, 1, 2)$ and $j = 1, 2$ and hence is following:

$$\begin{aligned} V'_0(x) &= V_0(x) - \frac{1}{2}\beta = \frac{\beta^2 x^2 - \beta}{2(1-x^2)}, \\ V'_1(x) &= \frac{\beta^2 x^2 + \beta}{2(1-x^2)}, \\ V'_2(x) &= V'_0(x) = \frac{\beta^2 x^2 - \beta}{2(1-x^2)} \end{aligned} \tag{33}$$

because $\hat{H}'_2 = \hat{H}'_0$ that means of $A_1^+ A_1 = A_2 A_2^+$ and then,

$$W_1 = \frac{\beta x}{\sqrt{1-x^2}}, \qquad W_2 = \frac{-\beta x}{\sqrt{1-x^2}}, \tag{34}$$

$$\hat{H}' = \hat{H}'_0 \oplus \hat{H}'_1 \oplus \hat{H}'_2 = \begin{pmatrix} \hat{H}'_0 & 0 & 0 \\ 0 & \hat{H}'_1 & 0 \\ 0 & 0 & \hat{H}'_2 \end{pmatrix} = \begin{pmatrix} A_1^+ A_1 & 0 & 0 \\ 0 & A_1 A_1^+ & 0 \\ 0 & 0 & A_1^+ A_1 \end{pmatrix}, \tag{35}$$

$$Q_1^- = \sqrt{2}\begin{pmatrix} 0 & A_1^+ & 0 \\ 0 & 0 & A_1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Q_1^+ = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ A_1 & 0 & 0 \\ 0 & A_1^+ & 0 \end{pmatrix}, \tag{36}$$

$$\begin{aligned} Q_2^- &= \sqrt{2}\begin{pmatrix} 0 & -A_2 & 0 \\ 0 & 0 & A_2^+ \\ 0 & 0 & 0 \end{pmatrix} = \sqrt{2}\begin{pmatrix} 0 & A_1^+ & 0 \\ 0 & 0 & -A_1 \\ 0 & 0 & 0 \end{pmatrix}, \\ Q_2^+ &= \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ -A_2^+ & 0 & 0 \\ 0 & A_2 & 0 \end{pmatrix} = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ A_1 & 0 & 0 \\ 0 & -A_1^+ & 0 \end{pmatrix}. \end{aligned} \tag{37}$$

Because $\hat{H}'_2 = \hat{H}'_0$ and $c = 0$, higher order PSUSY such as $P = 3$ is obtained to the following set of equation:

$$\begin{aligned} \hat{H}'_0 &= \frac{1}{2} Q_1^- Q_1^+ \\ &= -\frac{1}{2}(1-x^2)\frac{d^2}{dx^2} + \frac{1}{2}x\frac{d}{dx} + \frac{1}{2}\beta(\beta-1)\left(\frac{x^2}{1-x^2}\right) - \frac{1}{2}\beta, \\ \hat{H}'_1 &= \frac{1}{2} Q_1^+ Q_1^- = \frac{1}{2} Q_2^- Q_2^+ \\ &= -\frac{1}{2}(1-x^2)\frac{d^2}{dx^2} + \frac{1}{2}x\frac{d}{dx} + \frac{1}{2}\beta(\beta+1)\left(\frac{x^2}{1-x^2}\right) + \frac{1}{2}\beta, \\ \hat{H}'_2 &= \frac{1}{2} Q_2^+ Q_2^- = \frac{1}{2} Q_3^- Q_3^+ = \frac{1}{2} Q_1^- Q_1^+ = \hat{H}'_0 \\ &= -\frac{1}{2}(1-x^2)\frac{d^2}{dx^2} + \frac{1}{2}x\frac{d}{dx} + \frac{1}{2}\beta(\beta-1)\left(\frac{x^2}{1-x^2}\right) - \frac{1}{2}\beta, \\ \hat{H}'_3 &= \frac{1}{2} Q_3^+ Q_3^- = \frac{1}{2} Q_1^+ Q_1^- = \hat{H}'_1 \\ &= -\frac{1}{2}(1-x^2)\frac{d^2}{dx^2} + \frac{1}{2}x\frac{d}{dx} + \frac{1}{2}\beta(\beta+1)\left(\frac{x^2}{1-x^2}\right) + \frac{1}{2}\beta. \end{aligned} \tag{38}$$

On also can use this method to obtain other partners, supercharge and higher order parasupersymmetries. We do not proceed in this direction here.

6 Conclusion

In this paper after comparing nonlinear oscillator on the flat space with isotropic oscillator on the curved space we obtained partner Hamiltonian and super-charges by use of factorization method.

In future, one can compare other oscillators in flat and curved space in order to understand the relation between the two space. Also, by studying the exact configuration of parafermionic algebra, one can look for other examples of higher order PSUSY to discover the hidden symmetries of our world.

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